INTERNATIONAL MATHEMATICS
# TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 40, Northern Autumn 2018 (A Level)



**Note:** Each contestant is credited with the largest sum of points obtained for three problems.

**1.** There are 2018 people living on an island. Each person is one of: a knight, a knave, or a neither-knight-nor-knave. A knight always tells the truth, and a knave always lies. A neither-knight-nor-knave answers as the majority of people answered before him, or randomly, in the case that the numbers of *"Yes"* and *"No"* answers are equal. Everyone on the island knows which of the three possibilities each person is. One day all 2018 inhabitants of the island were arranged in a line and each in turn answered *"Yes"* or *"No"* to the same question:

*Are there more knights than knaves on the island?*

The total number of *"Yes"* answers was 1009 and everyone heard all the previous answers. Determine the maximum possible number of neither-knight-nor-knave people among the inhabitants of the island. (5 points)

**2.** In non-isosceles acute-angled triangle $ABC$, $AH_a$ and $BH_b$ are altitudes, and $O$ is the circumcentre. Suppose that points $X$ and $Y$ are symmetric to the points $H_a$ and $H_b$ with respect to the midpoints of the sides $BC$ and $CA$, respectively. Prove that the line $CO$ bisects the line segment $XY$. (7 points)

**3.** Prove that

(a) any integer of the form $3k - 2$, where $k$ is an integer, can be represented as the sum of a perfect square and two perfect cubes of some integers. (6 points)

(b) any integer can be represented as the sum of a perfect square and three perfect cubes of some integers. (2 points)

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**4.** There are a finite number of squares of an infinite grid plane that are coloured black. All other squares are white. Consider a paper polygon $P$ placed on the grid plane with its sides along the grid lines, that contains at least two squares. $P$ can be moved (but without being rotated) in any direction and any distance so long as its sides are along the grid lines in the new position. After a move, if exactly one square covered by $P$ is white, then the colouring of that square is changed to black. Prove that there exists a white square which will never be coloured black, no matter how many times $P$ is moved, according to the rules above. (8 points)

**5.** The three medians of a triangle divide its angles into six angles. Let $k$ be the number of these six angles whose size is greater than $30^\circ$. What is the greatest possible value of $k$? (8 points)

**6.** An infinite number of points with positive integer coordinates are chosen on a number line. When a wheel rolls along the number line over the chosen points, each leaves a dot trace on the wheel. Prove that one can choose a real number $R$ such that if a wheel of radius $R$ starts rolling along the number line from 0, then each arc of size $1^\circ$ of the wheel will contain at least one dot trace of a chosen point. (9 points)

**7.** John and Karl play the following game. There are $n > 1$ towns, each populated by the same number of people. At the start of the game each person in each town has one coin (all coins are identical). On his turn, John chooses one person in each town; then Karl redistributes the coins of the chosen people among them, so that each person has a different number of coins to what they had immediately before the redistribution. John wins if at some moment there is at least one person in each town with no coins. Prove that John can always win, no matter how Karl plays, if in each town there are exactly

(a) $2n$ people. (10 points)

(b) $2n - 1$ people. (4 points)